Discrete Differential Geometry

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6 Discrete Surfaces

6.1 Smooth Manifold

Definition 6.1. A smooth *n-manifold* is a set M with a family $\{M_i\}_{i\in I}$ of subsets $M_i \subset M$ such that:

1. $\bigcup_i M_i = M$;
2. for all $i \in I$ there exists an injective map $\phi : M_i \to \mathbb{R}^n$ with $\phi(M_i)$ open in $\mathbb{R}^n$; and
3. whenever $M_i \cap M_j \neq \emptyset$, then $\phi_j \circ \phi_i^{-1} : \phi_i(M_i \cap M_j) \to \mathbb{R}^n$ is smooth and $\phi_i(M_i \cap M_j)$ is open in $\mathbb{R}^n$.

The topology of M is defined by:

A subset $O \subset M$ is open, if $\phi_i(O \cap M_i)$ is open in $\mathbb{R}^n$ for all $i \in I$.

6.2 Discrete Surfaces

Definition 6.2. A *surface* is a two-dimensional manifold (with boundary).

Definition 6.3. A *cell decomposition* of a surface M is a covering of M by a set $T = \{U_i\}$ of pairwise disjoint 0-, 1-, and 2-cells so that

1. For any n-dimensional cell $U \in T$, there exists a continuous map $\phi : \overline{D^n} \to M$ with $\phi : D^n \to U$ homeomorphic and $\phi : \partial D^n \to$a union of at most $(n-1)$-dimensional cells.
2. (Finite Closure) For any $U \in T$, its closure $\overline{U}$ intersects finitely many other cells.
3. (Weak Topology) $A \subseteq M$ is closed iff $A \cap \overline{U}$ is closed for all $U \in T$.

Definition 6.4. A *polyhedral surface* in $\mathbb{R}^n(\mathbb{R}^3)$ is an (immersed) two-dimensional submanifold of $\mathbb{R}^n$ with a cell decomposition whose edges are intervals of straight lines and faces are planar. Polyhedral surfaces are intuitively surfaces gluing together polygons along edges. These definitions can be generalized (eg, the faces may not necessarily be planar) or specialized (by including more structures, eg, normals, tangent planes).

There is no canonical definition of a *discrete surface*. We will call the combinatorial data $\mathbb{S} = (M, T)$ (cell decomposition of a surface) an *abstract discrete surface* and a map $f : \mathbb{S} \to \mathbb{R}^n$ its *geometric realization*. Sometimes geometric realizations will have no faces.

Considering polyhedral surfaces intrinsically, one describes them as a pair (M, d) of a 2-manifold M with a polyhedral metric d. (M, d) is called a *piecewise flat surface*.

Definition 6.5. A cell decomposition of a surface is called *regular* if the maps $\phi : \overline{D^n} \to M$ are homeomorphisms.

A regular cell decomposition is called *strongly regular* if for any two cells U_i, U_j, we have that either:

- $\overline{U_i} \cap \overline{U_j} = \emptyset$, or
- $\overline{U_i} \cap \overline{U_j}$ consists of at most the closure of one 1-cell (an edge).

A *convex polyhedron* is a strongly regular cell-decomposition of S^2. Let H be a polytope. H is convex iff $\forall P, Q \in H, \overline{PQ} \subset H$.

Definition 6.6. The combination $|V| - |E| + |F| = \chi$ is called the *Euler characteristic* of the surface with V the set of vertices (ie 0-cells), E the set of edges (ie 1-cells), and F the set of faces (ie 2-cells). The Euler characteristic is independent of the cell decomposition (and only depends on the surface). Compact orientable surfaces that are homeomorphic to a sphere with g handles (surfaces of genus g) have an Euler characteristic $\chi = 2 - 2g$.

6.3 Voronoi and Delaunay Tesselations

6.3.1 Voronoi and Delaunay Tesselations of $\mathbb{R}^2$

Definition 6.7. Let $V = \{P_1, \ldots, P_n\} \subset \mathbb{R}^2$ be a set of points. A *Voronoi region* (2-cell) is

$$W_i = \{P \in \mathbb{R}^2 \mid |PP_i| < |PP_j|, \ \forall j \neq i\}.$$

Voronoi cells

$$H_{ij} = \{P \in \mathbb{R}^2 \mid |PP_i| < |PP_j|\}$$

are convex polygons. $W_i = \bigcap_i H_{ij}$ are intersections of half planes.

$\bigcup_i \overline{W_i}$ is called a *Voronoi tesselation* of $\mathbb{R}^2$ (for V). A *tesselation* is a cell decomposition with polygonal cells. The edges of Voronoi tesselations are

$$\{P \in \mathbb{R}^2 \mid \exists i, j, \ |PP_i| = |PP_j| < |PP_k| \forall k \neq i, j\}$$

The vertices of Voronoi tesslations are

$$\{P \in \mathbb{R}^2 \mid \exists i, j, k, \ |PP_i| = |PP_j| = |PP_k| < |PP_m| \forall m\}$$

Voronoi tesselations of $\mathbb{R}^2$ are strongly regular (because they are tesselations with polygons).

Let Q be a vertex of a Voronoi tesslation and $l_Q := |QP_i| = |QP_j| = |QP_k|$ be the distance of Q from its closest neighboring points.

$$D_Q = \{P \in \mathbb{R}^2 \mid |PQ| < l_Q\}$$

is the disc centered at Q containing no pts of V. Its closure contains at least 3 points of V. Define H_Q as the convex hull of the set of points that generate the Voronoi cells meeting at Q. $\bigcup_Q H_Q (= \mathbb{R}^2)$ is called a *Delaunay tesselation* of $\mathbb{R}^2$.

Properties:

- Voronoi and Delaunay tesselations are dual to each other and the corresponding edges are orthogonal.
- (Empty Disc Property) All faces of Delaunay tesselations are convex circular polygons. The corresponding Delaunay discs D_Q, with $\overline{D_Q} \supset H_Q$, contain no vertices from V.
- Delaunay tesselations can be defined by the empty disc property without referring to the Voronoi cells.
- For a given $V \subset \mathbb{R}^2$, the Voronoi and Delaunay tesselations exist and are unique. See 6.9.

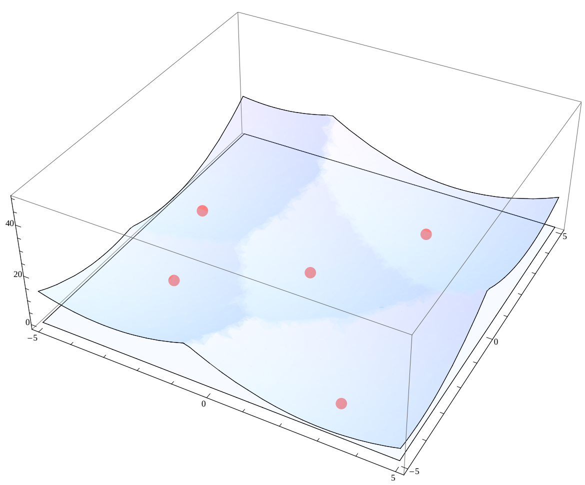
Definition 6.8. A tesselation with the vertices V in a plane is called Delaunay if it possesses the empty disc property, ie, its faces are circular convex polygons and the open discs of these polygons contain no vertices of V. An edge is called Delaunay if two faces sharing this edge do not have vertices in the interior of their discs.

6.3.2 Paraboloid Construction

Let $V = \{1, \ldots, i, \ldots, n\}$ be a set of points in $\mathbb{R}^2$. Introduce the paraboloids

$$\begin{aligned} q_i : \mathbb{R}^2 &\to \mathbb{R} \\ x &\mapsto |x - i|^2 \end{aligned}$$

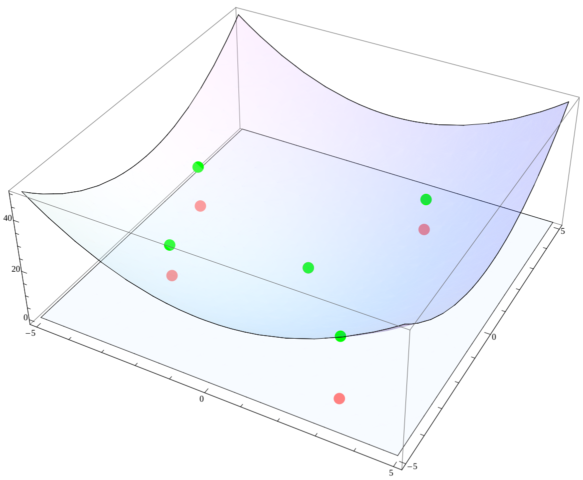
Let $q_{\min}(x) := \min_i q_i(x)$. The graph of $q_{\min}$ projected to $\mathbb{R}^2$ gives the Voronoi tesselation of $\mathbb{R}^2$ with V. The Voronoi 2-cells are the smoothness domains of $q_{\min}$.

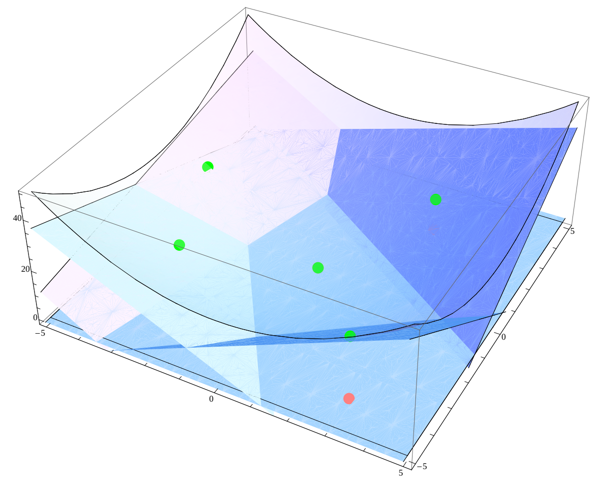


Let

$$\hat{q}(x) := q_0(x) - q_{\min}(x) = |x|^2 - |x - i|^2 = 2\langle x, i \rangle_{\mathbb{R}^2} - |i|^2$$

$\hat{q}$ is the tangent plane at $(i, |i|^2)$. $\hat{q}$ is a convex function. q_0 can be centered at any point.





We get a convex polyhedral surface touching the paraboloid $q_0(x)$ at the points $(i, |i|^2)$. Its projection to $\mathbb{R}^2$ is a Voronoi tesselation.

Let f be a Dealunay cell. Consider the paraboloid

$$y = p_f(x) = |x-a|^2 - b^2$$

determined by $p_f(i) = p_f(j) = p_f(k) = 0$. Define $p : \mathbb{R}^2 \to \mathbb{R}$ by

$$p(x) = \min_f p_f(x)$$

Then $\hat{p}(x) := |x|^2 - p(x)$ is a convex piecewise linear function and $\hat{p}(i) = |i|^2$ for all $i \in V$. All vertices $(i, |i|^2)$ belong to the half space $y \geq |x|^2 - p_f(x)$. The graph of $\hat{p}(x)$ is a convex polygonal surface inscribed in the paraboloid $y = |x|^2$.

Theorem 6.9. Given a set of points $V = \{x_1, \ldots, x_n\} \subset \mathbb{R}^2$, there exist unique Voronoi and Delaunay tesselations of $(\mathbb{R}^2, V)$. The tesselations are dual to each other.

- The Delaunay vertices V are the generating points of the Voronoi tesselation.
- The Delaunay cells are circular convex polygons centered at Voronoi vertices.
- The corresponding Voronoi and Delaunay edges are othogonal.
- Lifted to the paraboloid $\mathcal{P} = \{(x, |x|^2)\}$, the Voronoi and Delaunay tesselations correspond to convex polyhedral surfaces touching $\mathcal{P}$ at $(x_i, |x_i|^2)$ and inscribed in $\mathcal{P}$ with the vertices $(x_i, |x_i|^2)$ respectively.

Corollary 6.10. A tesselation is Delaunay iff all its edges are Delaunay. All edges of the polyhedron are convex iff it is a convex polyhedron.

6.3.3 Voronoi and Delaunay Tesselations of Polyhedral Surfaces

Definition 6.11. Let (M, d) be a compact piecewise flat surface without boundary and $V \subset M$ be a finite set of points containing all cone points. Typically, $V = \{P_1, \ldots, P_n\}$ is the set of the vertices of S (conical singularities of (M, d). A Voronoi cell is

$$W_i = \{P \in S \mid d(P, P_i) < d(P, P_j) \forall j \neq i\}$$

- Voronoi 2-cell = open disc
- Voronoi 1-cell = interior points of Voronoi edges

  Voronoi edges = $\{P \in S \mid \exists$ exactly two shortest geodesics to points of $V\}$, ie, there exists i, j such that $d(P_i, P) = d(P_j, P)$ with $P_i, P_j \in V$ (may be same point).

- Voronoi 0-cell = Voronoi vertices

  Voronoi vertices = $\{P \in S \mid \exists$ at least three shortest geodesics to points of $V\}$.

Voronoi 2-cells are empty immersed discs centered at Voronoi vertices. A Delaunay tesselation can be found by maximizing the immersed empty discs.

The *Delaunay tesselation* of $(M.d)$ with the vertex set V is the cell decomposition with the following 2-cells:

> A subset $C \subset M$ is a closed 2-cell if there exists an isometrically immersed empty disc $\phi : \overline{D} \to M$ such that $\phi_{-1}(V)$ contains more than 2 points and $C = \phi$(convex hull of $\phi^{-1}(V)$).

Theorem 6.12. For any piecewise flat surface with the vertex set V, there exists a unique Delaunay triangulation. A tesselation is Delaunay iff all its edges are Delaunay.

Note: Delaunay triangulations of polyhedral surfaces are not necessarily regular cell decompositions.

6.4 Curvatures of Polyhedral Surfaces

Smooth Gaussian Curvature

Definition 6.13. The definition of *Gaussian curvature* on a manifold is

$$K_p = \lim_{\epsilon \to 0} \frac{A(N(U_\epsilon))}{A(U_\epsilon)}$$

where $A(U_\epsilon)$ is an orientable area.

Discrete Gaussian Curvature

Definition 6.14. The angle defect $K(P) = 2\pi - \sum \alpha_i$, where α_i's are the angles of the vertices of the triangles glued at P, is called the *Gaussian curvature* of a polyhedral surface at P. The *total Gaussian curvature* is defined as

$$K = \sum_{P \in V} K(P)$$

Proof. Let $e_i \in E$ be the edge touching P and the intersection of the faces $f_{i-1}, f_i \in F$. Let N_{i-1}, N_i be the normal vectors to f_{i-1}, f_i respectively. Consider the plane spanned by N_{i-1}, N_i. Then, we know that e_i must be orthogonal to □

Let $S \subset \mathbb{R}^n$ be a polyhedral surface. A neighborhood of a point $P \in S$ on a

- face of S is isometric to a disc.
- edge of S is isometric to a disc.
- vertex of S is isometric to a cone.

Isometric deformations of polyhedral surfaces preserve the polyhedral metric and thus preserves the underlying piecewise flat surface (M, d). The Gaussian curvature is an intrinsic (has to do with the metric) curvature.

Theorem 6.15. The Gaussian curvature of a polyhedral surface is preserved by isometric deformations.

Theorem 6.16. (Gauss-Bonet Theorem) The total Gaussian curvature of a compact polyhedral surface S is given by

$$K(S) = \sum_{p \in V} K(p) = 2\pi\chi(S)$$

Proof.

$$\begin{aligned} K(S) = \sum_{p \in V} K(p) &= \sum_{p \in V} \left(2\pi - \sum_{\substack{\alpha_i \text{ angles at} \\ \text{vertex } p}} \alpha_i \right) = 2\pi|V| - \sum_{p \in V} \sum_{\substack{\alpha_i \text{ angles at} \\ \text{vertex } p}} \alpha_i \\ &= 2\pi|V| - \sum_{f \in F} \sum_{\substack{\alpha_i \text{ angles on} \\ \text{face } f}} \alpha_i = 2\pi|V| - \sum_{\substack{\tilde{\alpha_i} \text{ angles on} \\ \text{triangulated} S}} \tilde{\alpha_i} = 2\pi|V| - \pi|F| \\ &= 2\pi|V| - 2\pi(|E| - |F|) = 2\pi\chi(S) \end{aligned}$$

The last equality comes from the fact that we have a triangulation of a compact polyhedral surface, so $3|F| = 2|E|$. □

Definition 6.17. The *discrete mean curvature* of a polyhedral surface is a function on edges $e \in E$ given by

$$H(e) = \frac{1}{2}\theta(e)l(e)$$

where $l(e)$ is the length of e and $\theta(e)$ is the oriented angle between the normals of neighboring faces (positive in convex case and negative in concave case). The *total mean curvature* is

$$H(S) = \frac{1}{2}\sum_{e \in E} \theta(e)l(e)$$

Theorem 6.18. (Steiner's Theorem) Let P be a polyhdron and $\mathcal{P}_\rho$ be its parallel body

$$\mathcal{P}_\rho = \{p \in \mathbb{R}^3 \mid d(p, P) \leq \rho\}.$$

The volume of the convex body $\mathcal{P}_\rho$ is a cubic polynomial in ρ given by

$$V(\mathcal{P}_\rho) = V(P) + \rho A(\partial P) + \rho^2 H(\partial P) + \frac{4}{3}\pi\rho^3$$

where $A(\partial P)$ is the area of the body surface. The area of the body surface $\partial\mathcal{P}_\rho$ is a quadratic polynomial in ρ

$$A(\partial\mathcal{P}_\rho) = A(\partial P) + 2\rho H(\partial P) + 4\pi\rho^2$$

Proof. Let us find $A(\partial\mathcal{P}_\rho)$. Denote by $A(\partial P)$ the surface area of P. The contributions from the vertices $= 4\pi\rho^2$ (the surface area of a sphere). The cylindrical corners for an edge $e \in E$ will be $= 2\pi\rho l(e)\frac{\theta(e)}{2\pi} = \rho\theta(e)l(e) = 2\rho H(e)$. so

$$A(\partial\mathcal{P}_\rho) = A(\partial P) + 2\rho H(\partial P) + 4\pi\rho^2$$

Integrate this with respect to ρ to get the volume. □

Definition 6.19. The *normal shift* of a smooth surface M with the normal field N is defined as $M_\rho = M + \rho N$. If M is smooth, then M_ρ is also smooth for small ρ.

Theorem 6.20. Let M be a smooth surface and M_ρ its smooth normal shift for sufficiently small ρ. Then the area of M_ρ is a quadratic polynomial in ρ

$$A(M_\rho) = A(M) + 2\rho H(M) + K(M)\rho^2$$

where $H(M), K(M)$ are the total mean and Gaussian curvatures of M

$$H(M) = \int_M H \qquad K(M) = \int_M K$$

(This formula can be used as a definition of H, K.)