Discrete Differential Geometry

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3 Flow

Definition 3.1. A *flow* on a discrete curve $\gamma : I \to \mathbb{R}^n$ is a continuous deformation

$$\begin{aligned} \Gamma : I[\subset \mathbb{Z}] \times J[\subset \mathbb{R}] &\to \mathbb{R}^n \\ (k,t) &\mapsto \Gamma(k,t) \end{aligned}$$

by a vectorfield $v : I \times J \to \mathbb{R}^n$ along the curve. $\gamma^t(k) = \Gamma(k,t)$ and $v(k,t) = \frac{\partial}{\partial t}\Gamma(k,t)$.

Definition 3.2. A flow is called *tangent* if it is parallel to $\gamma_1 - \gamma_{\bar{1}}$ and preserves the arclength parametrization.

Proposition 3.3. The tangent flow of an arclength parametrized discrete curve is unique up to a multiplicative constant and is given by

$$\Gamma_t = \frac{T + T_{\bar{1}}}{1 + \langle T, T_{\bar{1}} \rangle}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$.

Proof. Let $\Gamma(\cdot, t) = \gamma^t : I \to \mathbb{R}^n$ be an arclength parametrized curve. The tangent vector is given by

$$\begin{aligned} T^t : I &\to \mathbb{R}^n \\ k &\mapsto \gamma^t(k+1) - \gamma^t(k) \end{aligned}$$

since γ is arclength parametrized, ie, the denominator is 1.

Notation:

$$\begin{aligned} \gamma^t(k) &=: \gamma & T^t(k) &= \gamma_1 - \gamma =: T \\ \gamma^t(k+1) &=: \gamma_1 & T^t(k+1) &= \gamma_2 - \gamma_1 =: T_1 \\ \gamma^t(k-1) &=: \gamma_{\bar{1}} & T^t(k-1) &= \gamma - \gamma_{\bar{1}} =: T_{\bar{1}} \end{aligned}$$

Note that $T + T_{\bar{1}} = \gamma_1 - \gamma_{\bar{1}}$.

For discrete arclength parametrized curves, the tangent vectors are of the form

$$v = \alpha(T, T_{\bar{1}})(\gamma_1 - \gamma_{\bar{1}})$$

where α is some function of two arguments.

We know that Γ is a tangent flow so by definition $v(k,t) = \frac{\partial}{\partial t}\Gamma(k,t) = \alpha(T,T_{\bar{1}})(\gamma_1 - \gamma_{\bar{1}}) = \alpha(T,T_{\bar{1}})(T + T_{\bar{1}})$, for all $k \in I$. Since $|T| = 1$ for all $k \in I$, $|T|^2 = 1$ and $\frac{d}{dt}|T|^2 = 0$ or

$$\frac{\partial}{\partial t}|T|^2 = \frac{\partial}{\partial t}\langle T,T\rangle = \langle \frac{\partial}{\partial t}T,T\rangle + \langle T, \frac{\partial}{\partial t}T\rangle = 2\langle T,T_t\rangle = 0$$

where

$$\begin{aligned} T_t &= \frac{\partial}{\partial t}(\gamma_1 - \gamma) = \frac{\partial}{\partial t}(\gamma_{k+1}^t - \gamma_k^t) = \frac{\partial}{\partial t}(\Gamma(k+1,t) - \Gamma(k,t)) \\ &= \frac{\partial}{\partial t}\Gamma(k+1,t) - \frac{d}{dt}\Gamma(k,t) \\ &= \alpha(T_1,T)(T_1 + T) - \alpha(T,T_{\bar{1}})(T + T_{\bar{1}}) \end{aligned}$$

Then

$$\begin{aligned} 0 &= \langle T,T_t\rangle \\ &= \langle T, (\alpha(T_1,T)(T_1+T) - \alpha(T,T_{\bar{1}})(T+T_{\bar{1}}))\rangle \\ &= \alpha(T_1,T)\langle T,(T_1+T)\rangle - \alpha(T,T_{\bar{1}})\langle T,(T+T_{\bar{1}})\rangle \\ &= \alpha(T_1,T)\Big(\langle T,T_1\rangle + \langle T,T\rangle\Big) - \alpha(T,T_{\bar{1}})\Big(\langle T,T\rangle + \langle T,T_{\bar{1}}\rangle\Big) \\ &= \alpha(T_1,T)\big(\langle T,T_1\rangle + 1\big) - \alpha(T,T_{\bar{1}})\big(1 + \langle T,T_{\bar{1}}\rangle\big) \end{aligned}$$

Thus

$$\alpha(T_1,T)\big(\langle T,T_1\rangle + 1\big) = \alpha(T,T_{\bar{1}})\big(1 + \langle T,T_{\bar{1}}\rangle\big)$$

and

$$\alpha(T,T_{\bar{1}})\big(\langle T_{\bar{1}},T\rangle + 1\big) = \alpha(T_{\bar{1}},T_{\bar{2}})\big(1 + \langle T_{\bar{1}},T_{\bar{2}}\rangle\big)$$

giving us

$$\alpha(T_1,T)\big(\langle T,T_1\rangle + 1\big) = \alpha(T_{\bar{1}},T_{\bar{2}})\big(1 + \langle T_{\bar{1}},T_{\bar{2}}\rangle\big) = c$$

for some constant c because the left hand side and right hand side are independent of the the variables they contain. So

$$\begin{aligned} c &= \alpha(T,T_{\bar{1}})\big(1 + \langle T,T_{\bar{1}}\rangle\big) \\ \alpha(T,T_{\bar{1}}) &= \frac{c}{1 + \langle T,T_{\bar{1}}\rangle} \end{aligned}$$

Finally,

$$v(k,t) = \frac{\partial}{\partial t}\Gamma(k,t) = \alpha(T,T_{\bar{1}})(T+T_{\bar{1}}) = \frac{T+T_{\bar{1}}}{1 + \langle T,T_{\bar{1}}\rangle} \cdot c$$

□

Lemma 3.4. Let $\gamma_{k-1}, \gamma_k, \gamma_{k+1}$ be 3 vertices of an arclength parametrized curve. Choose a vertex $\tilde{\gamma}_k$ of its Darboux transform infinitesimally close to γ_{k-1}.

$$\tilde{\gamma}_k = \gamma_{k-1} + \epsilon w + o(\epsilon), \qquad w \in \mathbb{C}.$$

Then, the next vertex of the Darboux transform $\tilde{\gamma}$ is

$$\tilde{\gamma}_{k+1} = \gamma_k + \epsilon V_k \langle w, T_{k-1} \rangle + o(\epsilon),$$

where

$$V_k = \frac{T_k + T_{k-1}}{1 + \langle T_k, T_{k-1} \rangle}$$

is the tangent flow of γ_k. In particular, if $w = T_{k-1}$,

$$\tilde{\gamma}_{k+1} = \gamma_k + \epsilon V_k + o(\epsilon).$$

Theorem 3.5. Darboux transformation of discrete arclength parametrized curves is *compatible* with its tangent flow.

4 Curvature

4.1 Smooth

Definition 4.1. Let $c : [a, b] \to \mathbb{R}^2$ be a regular arclength parametrized curve. The curvature $\kappa : [a, b] \to \mathbb{R}$ is given by

$$\kappa(s) := |\ddot{c}(s)|$$

The curvature of circles give the right intutition of the curvature of general curves. Namely the curvature of a general curve finds the "best approximating" circle. This circle is called the *osculating* circle and the inverse of its radius is the *curvature*.

4.2 Discrete

Definition 4.2. The *curvature* of $\gamma(t)$ is the inverse oriented radius of the osculating circle of $\gamma(t)$.

$$K = \frac{1}{R}.$$

One could select from a few different definitions of an osculating circle.

- A *vertex osculating circle* at $\gamma_k = \gamma$ is the circle through $\gamma_{\bar{1}}, \gamma, \gamma_1$. For arclength parametrized curves, K is bounded by 2.
- An *edge osculating circle* at $T_{\bar{1}}$ is the circle whose center is the intersection of the bisectors of γ_k and γ_{k-1} and tangent to $T_{k-1} = \gamma_k - \gamma_{k-1}$. The curvature is then defined by

$$|\gamma_k - \gamma_{k-1}| = R(\tan \frac{\phi_{k-1}}{2} + \tan \frac{\phi_k}{2})$$
$$K = \frac{\tan \frac{\phi_{k-1}}{2} + \tan \frac{\phi_k}{2}}{|\Delta \gamma_{k-1}|}$$

  where ϕ_k is the exterior angle between T_k, T_{k+1} at γ_k.

- An *edge osculating circle* at γ_k for an arclength parametrized curve is the circle touching T_{k-1}, T_k at their midpoints. The curvature is defined by

$$R \tan \frac{\phi_k}{2} = \frac{1}{2}$$
$$K = 2 \tan \frac{\phi_k}{2}$$

We will choose to use this last one.

5 Elastica

5.1 Smooth Elastica

Definition 5.1. *Elastica* are extremals (critical points) of the bending energy functional

$$E = \int_0^L K^2(s)\, ds$$

where $K(s) = |T'(s)|$ is the curvature at $\gamma(s)$.

Theorem 5.2. An arclength parametrized curve $\gamma : [0, L] \to \mathbb{R}^3$ is elastica iff its tangent vector $T : [0, L] \to S^2$ describes the evolution of the axis of a spherical pendulum.

5.2 Variational Calculus

The critical points of the functional

$$S = \int_0^L \mathcal{L}(q, q')\, ds$$

under variations preserving the constraints

$$F_i = \int_0^L f_i(q, q')\, ds = c_i \in R$$

are critical points of the functional

$$S_\lambda = S + \sum_i \lambda_i F_i,$$

where λ_i are Lagrange multipliers. In this case, the critical points of S_λ satisfy the constraints choosing λ_i.

The tranjectory $q(t)$ of a mechanical system with the potential energy U (depending on q) and the kinetic energy V (depending on q') is critical for the action functional

$$S = \int_0^T \mathcal{L}(q, q')\, dt = \int_0^T (V - U)\, dt$$

x is a critical point of a function $f(x)$ with respect to a constraint $h(x) = 0$ iff $\nabla f = -\lambda \nabla h$.

$$\nabla f + \sum_i \lambda_i \nabla h_i = 0$$

$$\nabla f_\lambda = 0 \qquad f_\lambda := f + \sum_i \lambda_i h_i$$

$$\frac{\partial}{\partial x_i} f_\lambda = \frac{\partial}{\partial x_i} f + \sum_i \lambda_i \frac{\partial}{\partial x_i} h_i = 0 \qquad \forall i = 1, \dots, n$$

If $M = \{x \mid h_j(x) \forall j = 1, \dots, k\}$ is compact, then critical points must exist.

5.3 Discrete Elastica

Definition 5.3. A discrete arclength parametrized curve $\gamma : T \to \mathbb{R}^3$ with tangent vector $T : I \to S^2$ is called *discrete elastica* if it is critical for the functional

$$S = \sum_i \log(1 + \frac{K_i^2}{4}) = \sum_i \log(1 + \langle T_i, T_{i+1} \rangle) \simeq \sum_i \log |T_i + T_{i+1}|$$

where $K_i = 2 \tan \frac{\phi_i}{2}$. [$\simeq$ means the functionals are equivalent.] The permissible variations should preserve the endpoints, $\gamma_0, \gamma_n \in \mathbb{R}^3$ and the endedges $T_0, T_{n-1} \in S^2$.

Theorem 5.4. (Euler-Lagrange equations for discrete elastica) A discrete arclength parametrized curve $\gamma : I \to \mathbb{R}^3$ is a discrete elastica iff there exist vectors $a, b \in \mathbb{R}^3$ such that

$$\frac{T_k \times T_{k-1}}{1 + \langle T_k, T_{k-1} \rangle} = a \times \gamma_k + b$$

where $T_k = \gamma_{k+1} - \gamma_k$.

Definition 5.5. The flow $\frac{\partial}{\partial t} \gamma_k = a \times \gamma_k + b$ on discrete arclength parametrized curves is called the *Heisenberg flow.*

Lemma 5.6. The Heisenberg flow is the only local flow in the binormal direction that commutes with the tangent flow.

Theorem 5.7. A discrete arclength parametrized curve is a discrete elastica iff the Heisenberg flow preserves its form, ie, under action of this flow, the curve is evolved by a Euclidean motion.

Definition 5.8. A discrete spherical pendulum is a map $T : \mathbb{Z} \to S^2$ with the Lagrangian

$$\mathcal{L} = \sum_k \log(1 + \langle T_k, T_{k-1} \rangle) - \langle a, T_k \rangle$$

$\log(1 + \langle T_k, T_{k-1} \rangle)$ is the kinetic and $\langle a, T_k \rangle$ potential energy of the pendulum.

Theorem 5.9. A discrete arclength parametrized curve $\gamma : I \to \mathbb{R}^3$ is a discrete elastica iff its tangent vector $T : I \to S^2$ describes the evolution of a discrete spherical pendulum.