# Discrete Differential Geometry

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# 7 Minimal Surfaces

## 7.1 Laplace Operator on Graphs

### 7.1.1 Smooth theory

Let $\Omega \subset \mathbb{R}^n$ be open. We define the Laplace operator as

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

A function $f : \Omega \to \mathbb{R}$ is called harmonic iff $\Delta f = 0$.

In a Dirichlet boundary value problem, we want to find $f$ such that $\Delta f = 0$ in $\Omega$ and $f = g$ on $\partial\Omega$.

The Dirichlet Energy is defined as

$$E(f) = \frac{1}{2} \int_\Omega |\text{grad } f|^2 \, dA$$

where $dA$ is the area functional and $\nabla f = \text{grad } f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$, giving us

$$|\nabla f|^2 = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right)^2 .$$

Let $V = \{f : \Omega \to \mathbb{R} \mid f|_{\partial\Omega} = g\}$ and $V_0 = \{f : \Omega \to \mathbb{R} \mid f|_{\partial\Omega} = 0\}$. Here $f \in C^2(\Omega)$. $f \in V, \tilde{f} \in V$ iff $\tilde{f} - f = \phi \in V_0$.

Consider a variation $f + t\phi \in V_0$

$$\begin{aligned} \frac{d}{dt} E(f + t\phi)|_{t=0} &= \frac{1}{2} \frac{d}{dt}|_{t=0} \int_\Omega |\nabla f + t\nabla\phi|^2 \, dA \\ &= \frac{d}{dt}|_{t=0} \frac{1}{2} \int_\Omega |\nabla f|^2 + 2t\langle \nabla f, \nabla \phi \rangle + t^2 |\nabla\phi|^2 \, dA \\ &= \int_\Omega \langle \nabla f, \nabla\phi \rangle \, dA \\ &= -\int_\Omega \Delta f \cdot \phi \, dA + \int_{\partial\Omega} \langle \nabla f, n \rangle \cdot \phi \, da \end{aligned}$$

where $n$ is a normal and $da$ is the length of the measure of $\partial\Omega$. Then

$$\int_\Omega \Delta f \phi \, dA = 0 \ \forall \phi \in V_0 \iff \Delta f = 0$$

That is, $f$ is a solution to the Dirichlet boundary value problem iff it is a ciritical point of the Dirichlet energy.

### 7.1.2 On Graphs

**Definition 7.1.** Let $G$ be a finite graph with vertices $V$ and edges $E$ (no faces). Let $f : V(G) \to \mathbb{R}$ and $\nu : E(G) \to \mathbb{R}$.

$$(\Delta f)(x) = \sum_{e=[x,x_i]\in E} \nu(e)(f(x) - f(x_i))$$

is called the *Laplace operator* on the graph $G$ with the weights $\nu : E \to \mathbb{R}$ on the edges. One can assume $\nu : E \to \mathbb{R}_*$.

A function $f : V \to \mathbb{R}$ is called *harmonic* if $\Delta f = 0$.

The *Dirichlet Energy* on a graph is

$$E(f) = \frac{1}{4} \sum_{e=[x,x_i]\in E} \nu(e)(f(x) - f(x_i))^2$$

If $\nu : E \to \mathbb{R}_+$ then the Dirichlet energy is positively defined.

**Theorem 7.2.** Let $V_0 \subset V$ (analogue of $\partial\Omega$). $c : V_0 \to \mathbb{R}$ is given. Then $f : V \to \mathbb{R}$ is a critical point of the Dirichlet energy on

$$\mathcal{F}_{V_0,c} = \{f : V \to \mathbb{R} \mid f|_{V_0} = c|_{V_0}\}$$

only if $f|_{V\setminus V_0}$ is harmonic.

**Theorem 7.3.** Let all weights be positive and the graph $G$ connected. Then there exists a unique minimizer of the Dirichlet energy on $\mathcal{F}_{V_0,c}, V_0 \neq \emptyset$. This minimizer is the unique solution to the Dirichlet boundaray value problem $\Delta f|_{V\setminus V_0} = 0, f|_{V_0} = c$.

## 7.2 Dirichlet Energy of Piecewise Linear Maps and the Discrete Laplace Operator

**Theorem 7.4.** Let $S$ be a simplicial surface and $f$ be a piecewise linear (on faces of $S$) continuous function. Then its Dirichlet energy is

$$E(f) = \frac{1}{4} \sum_{e=[x_i,x_j]\in E} \nu(e)(f(x_i) - f(x_j))^2$$

where the weights are

$$\nu(e) = \begin{cases} \cot\alpha_{ij} + \cot\alpha_{ji} & e \text{ internal} \\ \cot\alpha_{ij} & e \text{ a boundary edge} \end{cases}$$

Here $\alpha_{ij}, \alpha_{ji}$ are angles opposite $e$.

**Definition 7.5.** The operator

$$(\Delta f)(x) = \sum_{e=[x,x_i]\in E} \nu(e)(f(x) - f(x_i))$$

with the weights from above is called the *discrete cotangent Laplace operator*.

**Theorem 7.6.** Let $x_i \in \mathbb{R}^n$ be a vertex of a simplicial surface in $\mathbb{R}^n$. Then the area gradient is equal to the cotangent Laplace operator.

**Corollary 7.7.** Let $S$ be a critical point for the area functional. Then the coordinate function is harmonic.

## 7.3 Discrete Laplace Beltrami Operator

**Lemma 7.8.** The discs of two neighboring triangles are empty (ie, the edge $e$ is Delaunay) iff $\cot\alpha + \cot\beta \geq 0$.

**Definition 7.9.** Let $(M, d)$ be a piecewise flat surface and $\tau$ its Delaunay tesselation. The discrete Laplace Beltrami operator of $(M, d)$ is defined by

$$(\Delta f)(x_i) = \frac{1}{2} \sum_{\substack{x_j \\ [x_i,x_j]\in \text{ edges of } \tau}} (\cot\alpha_{ij} + \cot\alpha_{ji})(f(x_i) - f(x_j))$$

where the sum is taken over al the edges of the Delaunay tesselation.
Properties

- Generically $\tau$ is a triangulation. If $\tau$ is not a triangulation, they are circular $n$-gons with $n > 3$.
- All weights are positive.
- Existence and uniqueness of harmonic functions (with appropriate boundary conditions).

## 7.4 Delaunay Triangulation and the Dirichlet Energy

**Lemma 7.10.** (Rippa's Lemma)

$$E_\Box - E_{\Box^*} = \frac{(U_0 - U_0^*)^2(r_1 + r_3)(r_2 + r_4)}{4\sin\theta\; r_1 r_2 r_3 r_4}(r_1 r_3 - r_2 r_4)$$

*Proof.*

$$\begin{aligned}
u_x &= \frac{u_1 - u_0}{||u_1 - u_0||} = \frac{u_1 - u_0}{r_1} \\
u_2 - u_0 &= (r_2\cos\theta, r_2\sin\theta) - (0,0) = r_2\cos\theta\; u_x + r_2\sin\theta\; u_y \\
&= r_2\cos\theta\left(\frac{u_1 - u_0}{r_1}\right) + r_2\sin\theta\; u_y \\
&= \frac{r_2}{r_1}\cos\theta(u_1 - u_0) + r_2\sin\theta\; u_y \\
u_y &= \frac{1}{r_2\sin\theta}\left((u_2 - u_0) - \frac{r_2}{r_1}\cos\theta(u_1 - u_0)\right)
\end{aligned}$$

$$
\begin{aligned}
E(\triangle_1) &= \frac{1}{2}|\nabla u|^2 A(\triangle_1) = \frac{1}{2}\left\langle \begin{matrix} u_x \\ u_y \end{matrix}, \begin{matrix} u_x \\ u_y \end{matrix} \right\rangle \left(\frac{1}{2} r_1 r_2 \sin\theta\right) \\
&= \frac{1}{4}(r_1 r_2 \sin\theta)(u_x^2 + u_y^2) \\
&= \frac{1}{4}(r_1 r_2 \sin\theta)\left[\left(\frac{u_1 - u_0}{r_1}\right)^2 + \left(\frac{1}{r_2 \sin\theta}\left((u_2 - u_0) - \frac{r_2}{r_1}\cos\theta(u_1 - u_0)\right)\right)^2\right] \\
&= \frac{1}{4}\left[\frac{r_2}{r_1}\sin\theta(u_1 - u_0)^2 + \frac{r_1}{r_2 \sin\theta}\left((u_2 - u_0)^2 + \left(\frac{r_2}{r_1}\cos\theta(u_1 - u_0)\right)^2\right.\right. \\
&\qquad \left.\left. -2(u_2 - u_0)\left(\frac{r_2}{r_1}\cos\theta(u_1 - u_0)\right)\right)\right] \\
&= \frac{1}{4}\left[\frac{r_2}{r_1}\frac{\sin^2\theta}{\sin\theta}(u_1 - u_0)^2 + \frac{r_2}{r_1}\frac{\cos^2\theta}{\sin\theta}(u_1 - u_0)^2 + \frac{r_1}{r_2 \sin\theta}(u_2 - u_0)^2 - 2\frac{\cos\theta}{\sin\theta}(u_1 - u_0)(u_2 - u_0)\right] \\
&= \frac{1}{4\sin\theta}\left[\frac{r_2}{r_1}(u_1 - u_0)^2 + \frac{r_1}{r_2}(u_2 - u_0)^2 - 2\cos\theta(u_1 - u_0)(u_2 - u_0)\right] \\
E(\triangle_1^*) &= \frac{1}{4\sin\theta}\left[\frac{r_2}{r_1}(u_1 - u_0^*)^2 + \frac{r_1}{r_2}(u_2 - u_0^*)^2 - 2\cos\theta(u_1 - u_0^*)(u_2 - u_0^*)\right] \\
E(\triangle_1) - E(\triangle_1^*) &= \frac{1}{4\sin\theta}\left[\frac{r_2}{r_1}(u_0^2 - u_0^{*2} - 2u_1(u_0 - u_0^*)) + \frac{r_1}{r_2}(u_0^2 - u_0^{*2} - 2u_2(u_0 - u_0^*))\right. \\
&\qquad \left. -2\cos\theta(u_0^2 - u_0^{*2} - u_1(u_0 - u_0^*) - u_2(u_0 - u_0^*))\right] \\
&= \frac{1}{4\sin\theta}\left[\left(\frac{r_2}{r_1} + \frac{r_1}{r_2} - 2\cos\theta\right)(u_0^2 - u_0^{*2}) - \left(\frac{r_2}{r_1}2u_1 + \frac{r_1}{r_2}2u_2 - 2\cos\theta(u_1 + u_2)\right)(u_0 - u_0^*)\right] \\
E(\triangle_2) - E(\triangle_2^*) &= \frac{1}{4\sin(\pi - \theta)}\left[\left(\frac{r_3}{r_2} + \frac{r_2}{r_3} - 2\cos(\pi - \theta)\right)(u_0^2 - u_0^{*2})\right. \\
&\qquad \left. - \left(\frac{r_3}{r_2}2u_2 + \frac{r_2}{r_3}2u_3 - 2\cos(\pi - \theta)(u_2 + u_3)\right)(u_0 - u_0^*)\right] \\
&= \frac{1}{4\sin\theta}\left[\left(\frac{r_3}{r_2} + \frac{r_2}{r_3} + 2\cos\theta\right)(u_0^2 - u_0^{*2}) - \left(\frac{r_3}{r_2}2u_2 + \frac{r_2}{r_3}2u_3 + 2\cos\theta(u_2 + u_3)\right)(u_0 - u_0^*)\right] \\
E(\triangle_3) - E(\triangle_3^*) &= \frac{1}{4\sin\theta}\left[\left(\frac{r_4}{r_3} + \frac{r_3}{r_4} - 2\cos\theta\right)(u_0^2 - u_0^{*2}) - \left(\frac{r_4}{r_3}2u_3 + \frac{r_3}{r_4}2u_4 - 2\cos\theta(u_3 + u_4)\right)(u_0 - u_0^*)\right] \\
E(\triangle_4) - E(\triangle_4^*) &= \frac{1}{4\sin\theta}\left[\left(\frac{r_1}{r_4} + \frac{r_4}{r_1} + 2\cos\theta\right)(u_0^2 - u_0^{*2}) - \left(\frac{r_1}{r_4}2u_4 + \frac{r_1}{r_4}2u_1 + 2\cos\theta(u_4 + u_1)\right)(u_0 - u_0^*)\right]
\end{aligned}
$$

$$
\begin{aligned}
&E(\square) - E(\square^*) \\
&= (E(\triangle_1) + E(\triangle_2) + E(\triangle_3) + E(\triangle_4)) - (E(\triangle_1^*) + E(\triangle_2^*) + E(\triangle_3^*) + E(\triangle_4^*)) \\
&= \frac{(u_0 - u_0^*)}{4\sin\theta}\left[\left(\frac{r_2}{r_1} + \frac{r_1}{r_2} + \frac{r_3}{r_2} + \frac{r_2}{r_3} + \frac{r_4}{r_3} + \frac{r_3}{r_4} + \frac{r_1}{r_4} + \frac{r_4}{r_1}\right)(u_0 + u_0^*)\right. \\
&\qquad - 2\left(\frac{r_2}{r_1}u_1 + \frac{r_1}{r_2}u_2 + \frac{r_3}{r_2}u_2 + \frac{r_2}{r_3}u_3 + \frac{r_4}{r_3}u_3 + \frac{r_3}{r_4}u_4 + \frac{r_1}{r_4}u_4 + \frac{r_4}{r_1}u_1\right. \\
&\qquad \left.\left. - \cos\theta(u_1 + u_2) + \cos\theta(u_2 + u_3) - \cos\theta(u_3 + u_4) + \cos\theta(u_4 + u_1)\right)\right] \\
&= \frac{(u_0 - u_0^*)}{4\sin\theta}\left[\left(\frac{r_1 + r_3}{r_2} + \frac{r_1 + r_3}{r_4} + \frac{r_2 + r_4}{r_1} + \frac{r_2 + r_4}{r_1}\right)(u_0 + u_0^*)\right. \\
&\qquad \left. -2\left((r_2 + r_4)\left(\frac{u_1}{r_1} + \frac{u_3}{r_3}\right) + (r_1 + r_3)\left(\frac{u_2}{r_2} + \frac{u_4}{r_4}\right)\right)\right] \\
&= \frac{(u_0 - u_0^*)}{4\sin\theta}\left[\left(\frac{(r_1 + r_3)(r_2 + r_4)}{r_2 r_4} + \frac{(r_1 + r_3)(r_2 + r_4)}{r_1 r_3}\right)(u_0 + u_0^*)\right. \\
&\qquad \left. -2\left((r_2 + r_4)\left(\frac{r_3 u_1 + r_1 u_3}{r_1 r_3}\right) + (r_1 + r_3)\left(\frac{r_4 u_2 + r_2 u_4}{r_2 r_4}\right)\right)\right] \\
&= \frac{(u_0 - u_0^*)}{4\sin\theta}\left[(r_1 + r_3)(r_2 + r_4)\left(\frac{1}{r_2 r_4} + \frac{1}{r_1 r_3}\right)(u_0 + u_0^*)\right. \\
&\qquad \left. -2\left((r_2 + r_4)\left(u_0\,\frac{r_1 + r_3}{r_1 r_3}\right) + (r_1 + r_3)\left(u_0^*\,\frac{r_2 + r_4}{r_2 r_4}\right)\right)\right] \\
&= \frac{(u_0 - u_0^*)}{4\sin\theta}(r_1 + r_3)(r_2 + r_4)\left[\left(\frac{u_0 + u_0^*}{r_2 r_4} + \frac{u_0 + u_0^*}{r_1 r_3}\right) - 2\left(\frac{u_0}{r_1 r_3} + \frac{u_0^*}{r_2 r_4}\right)\right] \\
&= \frac{(u_0 - u_0^*)^2}{4\sin\theta}\,\frac{(r_1 + r_3)(r_2 + r_4)}{r_1 r_2 r_3 r_4}(r_1 r_3 - r_2 r_4)
\end{aligned}
$$

where we have

$$
\begin{aligned}
\frac{u_1 - u_0}{r_1} = (u_{\triangle_1})_x &= -(u_{\triangle_3})_x = \frac{u_0 - u_3}{r_3} \\
r_3 u_1 - r_3 u_0 &= r_1 u_0 - r_1 u_3 \\
u_0 &= \frac{r_3 u_1 + r_1 u_3}{r_1 r_3} \\
\frac{u_2 - u_0^*}{r_2} = (u_{\triangle_2})_x &= -(u_{\triangle_4})_x = \frac{u_0^* - u_4}{r_4} \\
u_0^* &= \frac{r_4 u_2 + r_2 u_4}{r_2 r_4}
\end{aligned}
$$

□

**Lemma 7.11.** (13) is a Delaunay edge of the quad (1234) iff $r_1 r_3 < r_2 r_4$. (1234) is circular iff $r_1 r_3 = r_2 r_4$. $E_\square - E_{\square^*} > 0$ iff (24) is Delaunay.

**Theorem 7.12.** Let $(S, d)$ be a piecewise flat surface and let $V \subset S$ be a set of marked points (containing all the cone points). Let $f : V \to \mathbb{R}$ be a function on marked points. For each triangulation $T$ with vertex set $V$, let $f_T : S \to \mathbb{R}$ be the piecewise linear interpolation of $f$ that is linear on the faces of the triangulation $T$. Then the minimum of the Dirichlet energy over all possible triangulations is attained on the Delaunay triangulation of $(S, d)$

$$\min_T \int |\nabla f_T|^2 = \int_S |\nabla f_{T_D}|^2$$

where $T_D$ is the Delaunay triangulation.

**Theorem 7.13.** A triangulation is Delaunay iff all its edges are Delaunay edges.

### 7.4.1 Harmonic Index of a triangulation

**Definition 7.14.**

$$h(\triangle) := \frac{a^2 + b^2 + c^2}{2A}$$

is called the *harmonic index of the triangle* $\triangle$.

$$h(T) := \sum_{\triangle \in T} h(\triangle)$$

is called the *harmonic index of a triangulation* $T$.

**Definition 7.15.** Let $\tau$ be the set of all triangulations of a piecewise flat surface on a given set of marked points and let $f : \tau \to \mathbb{R}$. We say that $f$ is *proper* if for any $M \in \mathbb{R}$, the number of triangulations $T \in \tau$ with $f(T) \leq M$ is finite.

**Proposition 7.16.** The harmonic index is a proper function.

### 7.4.2 Application to minimal surfaces

**Definition 7.17.** A polyhedral surface is called *minimal* if:

- (narrow definition) its triangulation is Delaunay and $\Delta f = 0$, where $\Delta$ is the Laplace Beltrami operator (which, in this case, coincides with the Laplace operator of polyhedral surfaces in $\mathbb{R}^3$).
- (wide definition) $\Delta f = 0$, where $\Delta$ is the Laplace Beltrami operator

In both cases, the weights are positive, thus the maximum principle is satisfied (ie, every vertex lies in the convex hull of its neighbors). In the first case, minimal surfaces are area minimizing.

Minimal surfaces $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ can be equivalently defined as:

- area minimizers
- $H = 0$ (mean curvature vanishes) $H = \Delta f$ where $\Delta$ is Laplace Beltrami operator of surface.

In the discrete case, we have seen that discrete surface $f$ is critical for the area functional iff $\Delta f = 0$ where $\Delta$ is the Laplace operator of the surface triangulation. Note: the weights of $\Delta$ are not necessarily positive.