Discrete Differential Geometry

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Introduction

The idea of discretizing concepts of classical differential geometry has been around for some time (Sauer, 1930's). But its development exploded over the last 15 years since advancements in technology opened up the possibility to experiment with different theories and elicited interest from industry to further investigate the potential applications in this field.

The main goal in discrete differential geometry is to formulate discrete equivalents of the geometric concepts and methods of classical differential geometry while preserving fundamental properties of the smooth theory. That is to say that one hopes to recover the smooth surface theory in the limit of refinements of the discretization but does not want to lose essential geometric information in the discretization.

This lecture series is based on a Discrete Differential Geometry course taught by Professor Dr. Alexander Bobenko at the Technische Universität at Berlin as part of the Berlin Mathematical School. These notes have been compiled from the lecture notes from the following courses:

- Professor Dr. Alexander Bobenko, Technische Universität Berlin, Geometry I (BMS Advanced Course), Wintersemester 2008/2009.

- Professor Dr. Alexander Bobenko, Technische Universität Berlin, Geometry II - Discrete Differential Geometry (BMS Basic Course), Sommersemester 2009.

- Professor Dr. Konrad Polthier, Freie Universität Berlin, Differential Geometry I (BMS Basic Course), Sommersemester 2010.

with some additional background information from the following books:

- Alexander Bobenko and Yuri Suris, Discrete differential geometry: Integrable structure.

- Isaac Chavel, Riemannian geometry: A modern introduction, 2nd edition.

- Manfredo do Carmo, Riemannian geometry.

- Timothy Gowers, The Princeton companion to mathematics.

- David Hilbert and S. Cohn-Vossen, Geometry and the imagination.

- John Stillwell, The four pillars of geometry.

1 Discrete Curves

Differential geometry explores the geometric properties of curves and surfaces by examining each of its points and their neighborhood. We can approximate the curve or surface in the neighborhood by comparing them to simple[1] curves or surfaces like lines and circles or planes and spheres.

To motivate the terms used to describe discrete curves, we will start by thinking about its smooth counterpart.

1.1 Smooth curves

Definition 1.1. Let $I = [a, b] \subset \mathbb{R}$ be an interval. A *parametrized curve* is a map[2]

$$\begin{aligned} c : I &\to \mathbb{R}^n \\ t &\mapsto c(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix} \in \mathbb{R}^n. \end{aligned}$$

Definition 1.2. The *tangent vector* of a curve c is

$$\frac{d}{dt} c(t) = \dot{c}(t) = \begin{pmatrix} \dot{c_1}(t) \\ \vdots \\ \dot{c_n}(t) \end{pmatrix}.$$

We want to study nice curves. We want curves to be differentiable. And we don't want curves to have sharp corners.[3] Such curves are said to be *regular*.

Definition 1.3. A curve is called *regular* if $|\dot{c}(t)| > 0$.

Regular curves have well-defined tangent lines and do not have sharp corners.

Definition 1.4. Let $c : I \to \mathbb{R}^n$ be a differentiable parametrized curve. The *length* of c is given by

$$L\left(c|_{[a,b]}\right) = \int_a^b |\dot{c}(t)| \, dt$$

where $|\dot{c}(t)| = \sqrt{\sum \left(\dot{c}_i(t)\right)^2}$.

Definition 1.5. A curve is said to be *arclength parametrized* or *parametrized by arclegnth* if $|\dot{c}(t)| = 1$ for all $t \in I$.

If c is an arclength parametrized curve, then

$$L\left(c|_{[a,b]}\right) = \int_a^b |\dot{c}(t)| \, dt = \int_a^b 1 \, dt = b - a.$$

This means that the length of curve is the same as the length of the interval.

[1] ie, easy

[2] The term *map* is used interchangeably with *continuous function*.

[3] In particular, we want to avoid situations where c maps an interval (a 1-manifold) to a space-filling curve (a 2-manifold). For example, fractals, Sierpiński curve.

1.2 Discrete curves

Definition 1.6. Let $I \subset \mathbb{Z}$ be a finite interval. A *discrete curve* in $\mathbb{R}^n$ is a map $\gamma : I \to \mathbb{R}^n$.
Notation: $\gamma_k := \gamma(k)$.

The same words used to talk about smooth curves are used to describe discrete curves. However, we are not able to use the same definitions since it does not make sense to talk about the differential of a curve, for example, when the curve contains nothing more than combinatorial data.

Definition 1.7. A discrete curve is called *regular* if any three of its successive points are different.

Definition 1.8. The *length* of a discrete curve $\gamma : I \to \mathbb{R}^n$ is given by

$$L(\gamma) = \sum_{k,k+1 \in I} |\gamma_{k+1} - \gamma_k| .$$

Definition 1.9. A discrete curve is called *arclength parametrized* if $|\gamma_{k+1} - \gamma_k| = 1$ for all $k, k+1 \in I$.

Definition 1.10. The *tangent vector* of γ at $k \in I$ is given by[4]

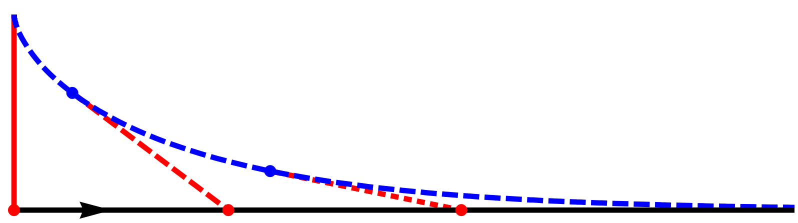
$$T_k := \frac{\gamma_{k+1} - \gamma_k}{|\gamma_{k+1} - \gamma_k|}.$$

2 Tractrix & Darboux Transform

2.1 Smooth

Let $\gamma : I \to \mathbb{R}^2$ be a smooth planar curve. Consider a point that moves along γ and pulls along an interval $[\gamma, \hat{\gamma}]$ so that

- the distance $|\hat{\gamma}(t) - \gamma(t)|$ is constant for all $t \in I$, and
- the velocity vector $\frac{d}{dt}\hat{\gamma}$ is parallel to $\hat{\gamma} - \gamma$.



Such a curve is called a *tractrix*.

Definition 2.1. A curve $\hat{\gamma}$ is called a *tractrix* of γ if $v := \hat{\gamma} - \gamma$ satisfies:

- $|v| = \text{const}$
- $v \parallel \hat{\gamma}'$

[4]If γ is arclength parametrized, then $T_k = \gamma_{k+1} - \gamma_k$.

Lemma 2.2. Let γ be an arclength parametrized curve and $\hat{\gamma}$ be its tractrix. Then the curve

$$\tilde{\gamma} := \gamma + 2v, \qquad \text{where } v = \hat{\gamma} - \gamma$$

is also arclength parametrized and $\hat{\gamma}$ is a tractrix of $\tilde{\gamma}$.

Proof. First, we begin by showing that $\tilde{\gamma}$ is arclength parametrized, ie, $|\tilde{\gamma}_k'| = 1\ \forall k \in I$.

$$\begin{aligned} |\tilde{\gamma}'|^2 &= \langle \gamma' + 2v', \gamma' + 2v' \rangle \\ &= \langle \gamma', \gamma' \rangle + 4\langle \gamma', v' \rangle + 4\langle v', v' \rangle \\ &= 1 + 4\langle \gamma' + v', v' \rangle = 1 + 4\langle \hat{\gamma}', v' \rangle \end{aligned}$$

Since $\hat{\gamma}$ is a tractrix of γ, we know that $v \parallel \hat{\gamma}'$. Also, $\langle v, v' \rangle = 0$. From this we know that $\langle \hat{\gamma}', v' \rangle = 0$. Thus $\tilde{\gamma}$ is arclength parametrized.

Now, we must show that $\hat{\gamma}$ is a tractrix of $\tilde{\gamma}$. Let $w := \hat{\gamma} - \tilde{\gamma}$. We want to show that (a) $|w|$ =const and (b) $w \parallel \hat{\gamma}'$.

(a) Note that we have

$$\tilde{\gamma} = \gamma + 2v = \gamma + 2(\hat{\gamma} - \gamma) = 2\hat{\gamma} - \gamma = \hat{\gamma} + v$$

Then,

$$w = \hat{\gamma} - \tilde{\gamma} = \hat{\gamma} - (\hat{\gamma} + v) = -v$$

So, $|w| = |-v| = |v|$ =const.

(b) Using the results from (a) again, we get

$$v \parallel \hat{\gamma}' \iff -v \parallel \hat{\gamma}' \iff w \parallel \hat{\gamma}'$$

so, w is parallel to $\hat{\gamma}$.

□

Definition 2.3. Two arclength parametrized curves $\gamma, \tilde{\gamma} : I \to \mathbb{R}^2$ are called *Darboux transforms* of each other if

- $|\gamma(t) - \tilde{\gamma}(t)| = \text{const}$ for all $t \in I$
- $\tilde{\gamma}$ is not a parallel translation[5] of γ.

Theorem 2.4. Let $\gamma : I \to \mathbb{R}^2$ be an arclength parametrized curve. Then the following statements are equivalent:

1) $\tilde{\gamma}$ is a Darboux transform of γ.

2) $\hat{\gamma} := \frac{1}{2}(\gamma + \tilde{\gamma})$ is a tractrix of γ (and of $\tilde{\gamma}$).

[5] The parallel translation of a curve is literally just translating the curve on the plane. More specifically, let $v \in S^1$ be a unit vector in $\mathbb{R}^2$. Then $\tilde{\gamma}(t) = \gamma(t) + c \cdot v$, where c is some constant.

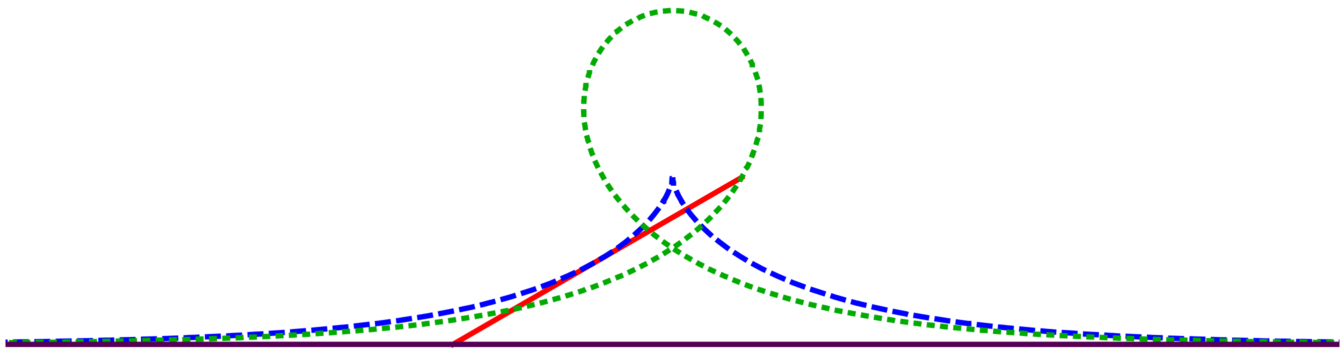


Figure 1: In this figure, the solid violet line is γ and the dotted green line is $\tilde{\gamma}$, which are Darboux transforms of each other. The blue dashed line is $\hat{\gamma}$, the tractrix of both $\gamma, \tilde{\gamma}$.

Proof. Let $v := \hat{\gamma} - \gamma$ and $w := \hat{\gamma} - \tilde{\gamma}$.

$(1 \to 2)$

For $\hat{\gamma}$ to be a tractrix of $\gamma, \tilde{\gamma}$, we need to show that

(a) $|v| =$const, $|w| =$const; and

(b) $v \parallel \hat{\gamma}'$, $w \parallel \hat{\gamma}'$

(a) We know that $\tilde{\gamma}$ is a Darboux transform of γ. So, $|\gamma - \tilde{\gamma}| =$const. Then

$$\begin{aligned} \hat{\gamma} &= \frac{1}{2}(\gamma + \tilde{\gamma}) \\ 2\hat{\gamma} &= \gamma + \tilde{\gamma} \\ \tilde{\gamma} &= 2\hat{\gamma} - \gamma \end{aligned}$$

and

$$\text{const} = |\gamma - \tilde{\gamma}| = |\gamma - (2\hat{\gamma} - \gamma)| = 2|\gamma - \hat{\gamma}|$$

and thus $|\gamma - \hat{\gamma}| = |-v| = |v| =$const. Similarly, $|w| =$const.

(b) Since $|\gamma - \tilde{\gamma}| =$const, we know that

$$\begin{aligned} \text{const} = c &= |\gamma - \tilde{\gamma}| \\ c^2 &= |\gamma - \tilde{\gamma}|^2 \\ 0 &= \frac{d}{dt}|\gamma(t) - \tilde{\gamma}(t)|^2 \\ &= \frac{d}{dt}\langle \gamma(t) - \tilde{\gamma}(t), \gamma(t) - \tilde{\gamma}(t) \rangle \\ &= 2\left\langle \frac{d}{dt}(\gamma(t) - \tilde{\gamma}(t)), \gamma(t) - \tilde{\gamma}(t) \right\rangle \\ &= 2\langle \gamma' - \tilde{\gamma}', \gamma - \tilde{\gamma} \rangle \\ \langle \gamma' - \tilde{\gamma}', \gamma - \tilde{\gamma} \rangle &= \langle \gamma', \gamma \rangle + \langle \tilde{\gamma}', \tilde{\gamma} \rangle - \Big(\langle \gamma', \tilde{\gamma} \rangle + \langle \tilde{\gamma}', \gamma \rangle \Big) = 0 \end{aligned}$$

We know that $\gamma, \tilde{\gamma}$ are arclength parametrized so

$$|\gamma| = 1 \quad \Rightarrow \quad \frac{d}{dt}|\gamma(t)|^2 = 0 \quad \Rightarrow \quad \langle \gamma', \gamma \rangle = 0 \ \& \ \langle \tilde{\gamma}', \tilde{\gamma} \rangle = 0$$

So we have

$$0 = \langle \gamma', \tilde{\gamma} \rangle + \langle \tilde{\gamma}', \gamma \rangle = \frac{d}{dt} \langle \gamma, \tilde{\gamma} \rangle \quad \Rightarrow \quad \langle \gamma, \tilde{\gamma} \rangle = \text{const}$$

($2 \to 1$) First notice that

$$\hat{\gamma} = \frac{1}{2}(\gamma + \tilde{\gamma}) \qquad \Rightarrow \qquad \tilde{\gamma} = 2\hat{\gamma} - \gamma = 2\hat{\gamma} - \gamma - \gamma + \gamma = \gamma + 2v$$

where $v = \hat{\gamma} - \gamma$. We know that γ is arclength parametrized and $\hat{\gamma}$ is its tractrix. By Lemma 2.2, we know that $\tilde{\gamma}$ is also arclength parametrized. We also know that $\hat{\gamma}$ is a tractrix of both γ and $\tilde{\gamma}$. Let $w = \hat{\gamma} - \tilde{\gamma}$ as before. Then, $v \parallel \hat{\gamma}' \parallel w$ and $\hat{\gamma}$ is on both v and w, so $\hat{\gamma}$ □

2.2 Discrete

Definition 2.5. Two discrete arclength parametrized curves $\gamma, \tilde{\gamma} : I \subset \mathbb{Z} \to \mathbb{R}^2$ are called *Darboux transforms* of each other if their corresponding points are at constant distance, $\|\gamma_k - \tilde{\gamma}_k\| = \text{const}$ for all k and $\gamma_k, \gamma_{k+1}, \tilde{\gamma}_{k+1}, \tilde{\gamma}_k$ is not a parallelogram. $(\gamma_k, \gamma_{k+1}, \tilde{\gamma}_{k+1}, \tilde{\gamma}_k)$ is called a *Darboux butterfly*.

3 Möbius Darboux Transform

Before we proceed, we will need some more terms. Some properties of these terms will also be listed without proof.

3.1 Möbius Transformation

Definition 3.1. A *Möbius transformation* or *linear fractional transformation* is a rational function of the form

$$\begin{aligned} L &: \mathbb{C} \to \mathbb{C} \\ z &\mapsto \frac{az+b}{cz+d} \qquad \text{where } a, b, c, d \in \mathbb{C},\ ad - bc \neq 0 \end{aligned}$$

This L can be identified with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$.

Note: There exists a bijection between linear fractional transformations and $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm I\}$ where $SL(2,\mathbb{C}) = \{A \in GL(2,\mathbb{C}) \mid \det A = 1\}$. This identification is a group isomorphism.

$$L_i \leftrightarrow [A_i] \in PSL(2,\mathbb{C}) \Rightarrow L_2 \circ L_1 \leftrightarrow [A_2 A_1]$$

Proposition 3.2. Möbius transformations map circles (and straight lines) to circles (and straight lines). They are compositions of reflections in hyperplanes[6] and inversions in hyperspheres.

Proposition 3.3. Möbius transformations are conformal.

[6]We treat hyperplanes as hyperspheres passing through ∞.

3.2 Cross Ratio

Definition 3.4. The *cross ratio* of four points in $\mathbb{C}$ is defined by

$$cr(z_1, z_2, z_3, z_4) := \frac{(z_1 - z_1)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

Proposition 3.5. Given any three points $p, q, r \in \mathbb{R}P^1$, any other point $x \in \mathbb{R}P^1$ is uniquely determined by its cross ratio with p, q, r, ie, $\exists!\, x$ such that $cr(p, q, r, x) = y$

Proposition 3.6. The cross ratio depends on the order of points.

Proposition 3.7. Four points $a, b, c, d \in \mathbb{C}$ are concircular (or collinear) iff their cross ratio is real. $cr < 0$ if the points are in order and $cr > 0$ if not in order around the circle.

Proposition 3.8. The cross ratio is invariant with respect to linear fractional transformations.

3.3 Möbius Darboux transform

Definition 3.9. Let $\gamma : I \to \mathbb{C}$ be a discrete curve and $\alpha_i \in \mathbb{R}$ (or $\mathbb{C}$) are associated to the edges $[\gamma_i, \gamma_{i+1}]$. A curve $\tilde{\gamma} : I \to \mathbb{C}$ is called a *Möbius Darboux transform* of γ with parameter $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) if

$$cr(\gamma_i, \gamma_{i+1}, \tilde{\gamma}_{i+1}, \tilde{\gamma}_i) = \frac{\alpha_i}{\lambda}.$$

$(\gamma_i, \gamma_{i+1}, \tilde{\gamma}_{i+1}, \tilde{\gamma}_i)$ is called a *Darboux butterfly.*

Theorem 3.10. Given a curve $\gamma : I \to \mathbb{C}, \lambda \in \mathbb{C}, \alpha_i$: edges $\to \mathbb{C}$ and a point $\tilde{\gamma}_0$, there exists a unique Möbius Darboux transform with these data.

Proof. Follows from Property 3.5. □

Theorem 3.11. (Closed Darboux Transforms) Let $\gamma : \mathbb{Z} \to \mathbb{C}$ be a periodic (ie, closed) discrete curve. Then for any $q \in \mathbb{C}$, there exist one, two, or infinitely many closed Darboux transformations $\tilde{\gamma} : \mathbb{Z} \to \mathbb{C}$.

Proof. When is $\tilde{\gamma}$ closed? ie $(\tilde{\gamma}_{i+n} = \tilde{\gamma}_i)$?

$$q = \frac{(\gamma_i - \gamma_{i+1})(\tilde{\gamma}_{i+1} - \tilde{\gamma}_i)}{(\gamma_{i+1} - \tilde{\gamma}_{i+1})(\tilde{\gamma}_i - \gamma_i)}$$

$$\begin{aligned} \tilde{\gamma}_i \mapsto \tilde{\gamma}_{i+1} &= \mathcal{M}_i(\tilde{\gamma}_i) \\ &= \frac{\tilde{\gamma}_i(\gamma_{i+1} - \gamma_i) + q\gamma_i(\gamma_{i+1} - \tilde{\gamma}_i)}{q(\gamma_i - \tilde{\gamma}_i) - (\gamma_i - \gamma_{i+1})} \end{aligned}$$

$$\begin{aligned} \tilde{\gamma}_n &= \mathcal{M}_{n-1}(\tilde{\gamma}_{n-1}) \\ &= \mathcal{M}_{n-1} \circ \mathcal{M}_{n-2}(\tilde{\gamma}_{n-2}) \\ &= \mathcal{M}_{n-1} \circ \mathcal{M}_{n-2} \circ \cdots \circ \mathcal{M}_0(\tilde{\gamma}_0) = \tilde{\gamma}_0 \end{aligned}$$

Since the composition of Möbius transformations is itself a Möbius transformation, we can set

$$\mathcal{M}_q := \mathcal{M}_{n-1} \circ \mathcal{M}_{n-2} \circ \cdots \circ \mathcal{M}_0$$

which will have the form

$$\mathcal{M}_q(z) = \frac{Az+B}{Cz+D},$$

where $A, B, C, D \in \mathbb{C}$ depends on γ and $q \in \mathbb{Q}$.

We want to find all $\tilde{\gamma}_0$, the Möbius Darboux transformations of γ, that will also be periodic, $\mathcal{M}_q(\tilde{\gamma}_0) = \tilde{\gamma}_0$. Then

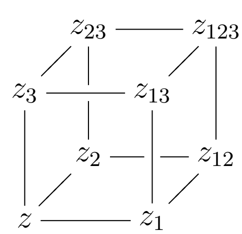
$$\begin{aligned} \mathcal{M}_q(\tilde{\gamma}_0) = \frac{A\tilde{\gamma}_0 + B}{C\tilde{\gamma}_0 + D} &= \tilde{\gamma}_0 \\ A\tilde{\gamma}_0 + B &= \tilde{\gamma}_0(C\tilde{\gamma}_0 + D) = C\tilde{\gamma}_0^2 + D\tilde{\gamma}_0 \\ C\,\tilde{\gamma}_0^2 + (D-A)\,\tilde{\gamma}_0 - B = 0 \end{aligned}$$

The solutions to this quadratic equation will give us $\tilde{\gamma}_0$ which in turn will give us the rest of $\tilde{\gamma}$. Since the equation is quadratic, we know that there will be one, two, or infinitely many[7] solutions.

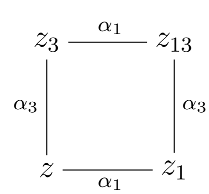
The set of all periodic Darboux transformations is associated to the algebraic curve $C = \{(q, \tilde{\gamma}_0) \in \mathbb{C}^2 \mid \mathcal{M}_q(\tilde{\gamma}_0) = \tilde{\gamma}_0\}$. □

The following Corollary 3.13 will be used in the next lecture.

Theorem 3.12. The cross ratio system is 3D-consistent (ie, the three possibly different values of z_{123} coincide for any choice of z, z_1, z_2, z_3).



Proof. Let us look at z, z_1, z_{13}, z_3.



[7] The parabola can intersect the x-axis once, twice or it can be the entire x-axis itself, thus having infinite solutions.

$$\begin{aligned}
cr(z, z_1, z_{13}, z_3) &= \frac{\alpha_1}{\alpha_3} \\
\frac{(z - z_1)}{(z_1 - z_{13})} \frac{(z_{13} - z_3)}{(z_3 - z)} &= \frac{\alpha_1}{\alpha_3} \\
(z_{13} - z_3) &= \frac{\alpha_1}{\alpha_3} \frac{(z_1 - z_{13})(z_3 - z)}{(z - z_1)} \\
(z_{13} - z_1) + (z_1 - z_3) &= -(z_{13} - z_1) \frac{\alpha_1}{\alpha_3} \frac{(z_3 - z)}{(z - z_1)} \\
(z_{13} - z_1) + (z_{13} - z_1) \frac{\alpha_1}{\alpha_3} \frac{(z_3 - z)}{(z - z_1)} &= -(z_1 - z_3) \\
(z_{13} - z_1) \left(1 + \frac{\alpha_1}{\alpha_3} \frac{(z_3 - z)}{(z - z_1)}\right) &= (z_3 - z_1) \\
(z_{13} - z_1) &= \frac{(z_3 - z_1)}{\left(1 + \frac{\alpha_1}{\alpha_3} \frac{(z_3 - z)}{(z - z_1)}\right)} \\
&= \frac{(z_3 - z) + (z - z_1)}{1 + \frac{\alpha_1}{\alpha_3} \frac{1}{(z - z_1)} (z_3 - z)} \\
&= \frac{1 \cdot (z_3 - z) + (z - z_1)}{\frac{\alpha_1}{\alpha_3} \frac{1}{z - z_1} (z_3 - z) + 1} \\
&=: L(z_1, z, \alpha_1, \alpha_3)[z_3 - z]
\end{aligned}$$

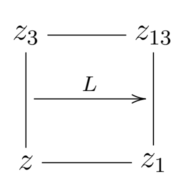
where $L(z_1, z, \alpha_1, \alpha_3)$ is a matrix defined by

$$L(w, v, \alpha, \beta) = \begin{pmatrix} 1 & v - w \\ \frac{\alpha}{\beta} \frac{1}{v - w} & 1 \end{pmatrix}, \qquad z, w \in \mathbb{C}, z \neq w, \alpha, \beta \in \mathbb{C}^*$$

whose associated linear fractional transformation is

$$L(w, v, \alpha, \beta)[z] = \frac{z + (v - w)}{\frac{\alpha}{\beta} \frac{1}{(v - w)} (z) + 1}.$$

This means that there is a Möbius transformation mapping $z_3 - z$ to $z_{13} - z_1$ and similarly for all edges on the combinatorial cube.



Going around the 3D cube, we get

$$\begin{aligned}
z_{123} - z_{12} &= L(z_{12}, z_1, \alpha_2, \alpha_3)[z_{13} - z_1] \\
&= L(z_{12}, z_1, \alpha_2, \alpha_3) L(z_1, z, \alpha_1, \alpha_3)[z_3 - z] \\
z_{123} - z_{12} &= L(z_{12}, z_2, \alpha_1, \alpha_3) L(z_2, z, \alpha_2, \alpha_3)[z_3 - z]
\end{aligned}$$

Claim:

$$cr(z, z_1, z_{12}, z_2) = \frac{\alpha_1}{\alpha_2} \iff L(z_{12}, z_1, \alpha_2, \alpha_3)L(z_1, z, \alpha_1, \alpha_3) = L(z_{12}, z_2, \alpha_1, \alpha_3)L(z_2, z, \alpha_2, \alpha_3)$$

The cross ratio: $cr(z, z_1, z_{12}, z_2) = \frac{\alpha_1}{\alpha_2}$ gives us the following relations:

$$\alpha_1 \frac{(z_1 - z_{12})}{(z - z_1)} = \alpha_2 \frac{(z_2 - z_{12})}{(z - z_2)}$$
$$\alpha_2 \frac{(z - z_1)}{(z_1 - z_{12})} = \alpha_1 \frac{(z - z_2)}{(z_2 - z_{12})}$$

Then we have:

$$\begin{aligned} L(z_{12}, z_1, \alpha_2, \alpha_3)L(z_1, z, \alpha_1, \alpha_3) &= \begin{pmatrix} 1 & z_1 - z_{12} \\ \frac{\alpha_2}{\alpha_3}\frac{1}{(z_1-z_{12})} & 1 \end{pmatrix} \begin{pmatrix} 1 & z - z_1 \\ \frac{\alpha_1}{\alpha_3}\frac{1}{(z-z_1)} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{\alpha_1}{\alpha_3}\frac{(z_1-z_{12})}{(z-z_1)} & z - z_{12} \\ \frac{\alpha_2}{\alpha_3}\frac{1}{(z_1-z_{12})} + \frac{\alpha_1}{\alpha_3}\frac{1}{(z-z_1)} & \frac{\alpha_2}{\alpha_3}\frac{(z-z_1)}{(z_1-z_{12})} + 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{\alpha_2}{\alpha_3}\frac{(z_2-z_{12})}{(z_2-z)} & z - z_{12} \\ \frac{\alpha_1}{\alpha_3}\frac{1}{(z_2-z_{12})} + \frac{\alpha_2}{\alpha_3}\frac{1}{(z-z_2)} & \frac{\alpha_1}{\alpha_3}\frac{(z-z_2)}{(z_2-z_{12})} + 1 \end{pmatrix} \\ &= L(z_{12}, z_2, \alpha_1, \alpha_3)L(z_2, z, \alpha_2, \alpha_3) \end{aligned}$$

Conversely, we have

$$L(z_{12}, z_1, \alpha_2, \alpha_3)L(z_1, z, \alpha_1, \alpha_3) = L(z_{12}, z_2, \alpha_1, \alpha_3)L(z_2, z, \alpha_2, \alpha_3)$$
$$\begin{pmatrix} 1 + \frac{\alpha_1}{\alpha_3}\frac{(z_1-z_{12})}{(z-z_1)} & z - z_{12} \\ \frac{\alpha_2}{\alpha_3}\frac{1}{(z_1-z_{12})} + \frac{\alpha_1}{\alpha_3}\frac{1}{(z-z_1)} & \frac{\alpha_2}{\alpha_3}\frac{(z-z_1)}{(z_1-z_{12})} + 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{\alpha_2}{\alpha_3}\frac{(z_2-z_{12})}{(z-z_2)} & z - z_{12} \\ \frac{\alpha_1}{\alpha_3}\frac{1}{(z_2-z_1)} + \frac{\alpha_2}{\alpha_3}\frac{1}{(z-z_2)} & \frac{\alpha_1}{\alpha_3}\frac{(z-z_2)}{(z_2-z_{12})} + 1 \end{pmatrix}$$

From position (1,1) we get the equality:

$$\begin{aligned} 1 + \frac{\alpha_1}{\alpha_3}\frac{(z_1 - z_{12})}{(z - z_1)} &= 1 + \frac{\alpha_2}{\alpha_3}\frac{(z_2 - z_{12})}{(z - z_2)} \\ \alpha_1 \frac{(z_1 - z_{12})}{(z - z_1)} &= \alpha_2 \frac{(z_2 - z_{12})}{(z - z_2)} \\ \frac{\alpha_1}{\alpha_2} &= \frac{(z - z_1)}{(z_1 - z_{12})}\frac{(z_2 - z_{12})}{(z - z_2)} = \frac{(z - z_1)}{(z_1 - z_{12})}\frac{(z_{12} - z_2)}{(z_2 - z)} \\ &= cr(z, z_1, z_{12}, z_2) \end{aligned}$$

We get the same conclusion from position (2,2) so the claim is proved.

No matter how we go around the combinatorial cube, we will obtain the same value of z_{123}, thus the cross-ratio system is 3D consistent. □

Corollary 3.13. Given a point z and its three neighbors $z_1, z_2, z_3 \in \mathbb{C}$, let z_{ij} be the Darboux tranforms associated to the faces of the 3D combinatorial cube. Then there exists a unique $z_{123} \in \mathbb{C}$ so that the faces

$$(z_1, z_{12}, z_{123}, z_{13})$$
$$(z_2, z_{23}, z_{123}, z_{12})$$
$$(z_3, z_{13}, z_{123}, z_{23})$$

are Darboux butterflies.

Proof. Choose $\alpha_i = \ell_i^2$ where $\ell = |z_i - z|$. Follows from Theorem 3.12. $\square$