Announcements

- M 11/28 from 3:10-4PM 204 Art
- W 11/30 6:10-8PM 6 Olson
- R 12/1 for 4:10-5PM 6 Wellman

MAT 21C REVIEW

10 Infinite Sequences and Series: 10.1-10.10

Sequences	Partial Sums of	...Series	...Power Series	...Taylor's Series
a_0	$s_0 =$	a_0	a_0	$f(0)$
a_1	$s_1 =$	$a_0 + a_1$	$a_0 + a_1 x$	$f(0) + f'(0)x$
a_2	$s_2 =$	$a_0 + a_1 + a_2$	$a_0 + a_1 x + a_2 x^2$	$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$
$\vdots$				
a_n	$s_n =$	$\sum_{k=0}^{n} a_k$	$\sum_{k=0}^{n} a_k x^k$	$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$
$\vdots$				

Important Concepts

- limits
- convergence
- intervals of convergence

12 Vectors and the Geometry of Space: 12.1-12.5

Vectors

- A vector is an ordered set of real numbers: $\vec{v} = \langle v_1, v_2 \rangle$ or $\vec{v} = \langle v_1, v_2, v_3 \rangle$
- The length or magnitude of $\vec{v}$ is $|\vec{v}| = \sqrt{v_1^2 + v_2^2}$ or $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- If $\vec{v} \neq \vec{0}$, the direction of $\vec{v}$ is the unit vector $\frac{\vec{v}}{|\vec{v}|}$.
- $\vec{v} = |\vec{v}| \dfrac{\vec{v}}{|\vec{v}|}$

Vector Operations

For vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and constant k

- Addition: $\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
- Scalar Multiplication: $k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$
- Dot Product: $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$ ($\Leftarrow$ a scalar!)
- Cross Product: $\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle$ ($\Leftarrow$ only in dimension 3!)

Properties of Vector Operations

- $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$, where θ is the acute angle between $\vec{u}$ and $\vec{v}$
- $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
- $\vec{u}$ and $\vec{v}$ are orthogonal when $\vec{u} \cdot \vec{v} = 0$
- The vector projection of $\vec{u}$ onto $\vec{v}$ is $\text{proj}_{\vec{v}}\, \vec{u} = \left(|\vec{u}|\cos\theta\right)\dfrac{\vec{v}}{|\vec{v}|} = \left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\right)\dfrac{\vec{v}}{|\vec{v}|} = \left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v}$
- The scalar component of $\vec{u}$ in the direction of $\vec{v}$ is $|\vec{u}|\cos\theta = \left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\right) = \vec{u} \cdot \dfrac{\vec{v}}{|\vec{v}|}$
- $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}|\sin\theta\, \vec{n}$, where $\vec{n}$ is the unit vector pointing in the normal direction
- $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ is the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$
- Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel when $\vec{u} \times \vec{v} = \vec{0}$
- Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel when $\vec{u} = k\vec{v}$ for some scalar $k \neq 0$.
- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

Lines and Planes in Space

- A vector equation for the line through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v}$ is $\vec{r}(t) = \vec{r}_0 + t\vec{v}$, where $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$ is the position vector of P_0.
- A vector equation for the plane through $P_0(x_0, y_0, z_0)$ normal to $\vec{v} = \langle v_1, v_2, v_3\rangle$ is
  $\vec{v} \cdot \overrightarrow{P_0P} = 0$ or
  $v_1(x - x_0) + v_2(y - y_0) + v_3(z - z_0) = 0$
- The distance from a point S to a line through P parallel to $\vec{v}$ is $d = \dfrac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|}$
- The distance from a point S to a plane through P with normal $\vec{v}$ is $d = \left|\overrightarrow{PS} \cdot \dfrac{\vec{v}}{|\vec{v}|}\right|$

13 Vector-Valued Functions and Motion in Space: 13.1-13.2

Curves in Space

- We can describe a curve in space as a collection of the positions in space traced by a particle's path.
- If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is the position vector of a particle moving along a smooth curve in space, then $\vec{v}(t) = \dfrac{d\vec{r}}{dt}$ is the particle's velocity vector and $\vec{a}(t) = \dfrac{d\vec{v}}{dt}$, when it exists, is the particle's acceleration.

Differentiation Rules

- Product Rule: $\dfrac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}\,'(t)\,\vec{v}(t) + \vec{u}(t)\,\vec{v}\,'(t)$
- Chain Rule: $\dfrac{d}{dt}[\vec{u}(f(t))] = f'(t)\,\vec{u}\,'(f(t))$
- When $|\vec{r}(t)| = c$ is constant, $\vec{r} \cdot \vec{r}\,' = 0$.

Initial Value Problem

- A differentiable function $\vec{\boldsymbol{R}}(t)$ is an antiderivative of a vector function $\vec{\boldsymbol{r}}$ if $\frac{d\vec{\boldsymbol{R}}}{dt} = \vec{\boldsymbol{r}}$. Then the indefinite integral of $\vec{\boldsymbol{r}}(t)$ is $\displaystyle\int \vec{\boldsymbol{r}}(t)\,dt = \vec{\boldsymbol{R}}(t) + \vec{\boldsymbol{C}}$, for some constant vector $\vec{\boldsymbol{C}}$.

> Differential Equation: $\dfrac{d}{dt}\vec{\boldsymbol{r}}(t) = \langle f(t), g(t), h(t) \rangle$
>
> Initial Condition: $\vec{\boldsymbol{r}}(0) = \vec{\boldsymbol{r}}_0 = \langle u_1, u_2, u_3 \rangle$
>
> Problem: Find $\vec{\boldsymbol{r}}(t)$.

1. Integrate differential equation:
$$\begin{aligned}\vec{\boldsymbol{r}}(t) &= \int \frac{d}{dt}\vec{\boldsymbol{r}}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle \\ &= \langle F(t)+c_1, G(t)+c_2, H(t)+c_3 \rangle = \langle F(t), G(t), H(t) \rangle + \langle c_1, c_2, c_3 \rangle\end{aligned}$$
where $F'(t) = f(t), G'(t) = g(t), H'(t) = h(t)$.
2. Use initial condition to solve for $\vec{\boldsymbol{C}} = \langle c_1, c_2, c_3 \rangle$:
$$\begin{gathered}\langle u_1, u_2, u_3 \rangle = \vec{\boldsymbol{r}}_0 = \vec{\boldsymbol{r}}(0) = \langle F(0), G(0), H(0) \rangle + \langle c_1, c_2, c_3 \rangle \\ \begin{aligned} u_1 &= F(0) + c_1 &\Rightarrow&\quad c_1 = u_1 - F(0) \\ u_2 &= G(0) + c_2 &\Rightarrow&\quad c_2 = u_2 - G(0) \\ u_3 &= H(0) + c_3 &\Rightarrow&\quad c_3 = u_3 - H(0)\end{aligned}\end{gathered}$$
3. Solution: $\vec{\boldsymbol{r}}(t) = \big\langle\, F(t) + u_1 - F(0),\ G(t) + u_2 - G(0),\ H(t) + u_3 - H(0) \,\big\rangle$

14 Partial Derivatives: 14.1-14.8

Partial Derivatives

- The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$
- $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$
- Chain Rule: $\dfrac{d}{dt} f(x(t), y(t)) = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

Implicit Differentiation

> Suppose $f(x, y) = 0$ and it is difficult to write y as a function of x.
>
> Problem: Find $\dfrac{dy}{dx}$.

1. Make sure to find f in the form $f(x, y) = 0$. (Everything has to be on same side of the equal sign.)
2. Compute f_x, f_y. Make sure $f_y \neq 0$.
3. Solution: $\dfrac{dy}{dx} = \dfrac{-f_x}{f_y}$

Directional Derivative and Gradient

- The directional derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\vec{\boldsymbol{u}} = \langle u_1, u_2 \rangle$ is
$$(D_{\vec{\boldsymbol{u}}}\, f)_{P_0} = \left.\frac{df}{dt}\right|_{\vec{\boldsymbol{u}}, P_0} = \lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$
provided the limit exists.
- The gradient vector of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$
- $D_{\vec{\boldsymbol{u}}}\, f = \nabla f \cdot \vec{\boldsymbol{u}}$
- The function f increases most rapidly in the direction of ∇f.
- The derivative along a path is $\frac{d}{dt} f(\vec{\boldsymbol{r}}(t)) = \nabla f(\vec{\boldsymbol{r}}(t)) \cdot \vec{\boldsymbol{r}}\,'(t)$
- If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\vec{\boldsymbol{r}} = x(t)\,\vec{\boldsymbol{i}} + y(t)\,\vec{\boldsymbol{j}}$, then $f(x(t), y(t)) = c$. Here $\vec{\boldsymbol{r}}(t)$ describes a level curve of f. Then ∇f is orthogonal to the tangent vector $\frac{d}{dt}\vec{\boldsymbol{r}}$. The tangent line to a level curve $f(x, y) = c$ at (x_0, y_0) is
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$
- The tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$. The tangent plane to a level surface $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is
$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$
The normal line of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$. The normal line to a level surface $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is
$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t$$
- The linearization of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function
$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Optimization

- Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then
  1. f has a *local maximum* at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$.
  2. f has a *local minimum* at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$.
  3. f has a *saddle point* at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$.
  4. *the test is inconclusive* at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$. In this case, we must find some other way to determine the behavior of f at (a, b).
- The method of Lagrange multipliers

> Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$.
>
> Problem: Find local min/max values of f subject to the constraint $g(x, y, z) = 0$ (if they exist).

Solution: Find values of x, y, z and λ that simultaneously satisfy the equations
$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0$$

- Lagrange multipliers with two constraints

Problem: Find extreme value of $f(x, y, z)$ whose variables are subject to two constraints

$$g_1(x, y, z) = 0 \qquad g_2(x, y, z) = 0$$

where g_1, g_2 are differentiable with ∇g_1 not parallel to ∇g_2.

Solution: Locate points $P(x, y, z)$ where f takes on its constrained extreme values. Find x, y, z, λ, μ that simultaneously satisfy

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \qquad g_1(x, y, z) = 0 \qquad \text{and} \qquad g_2(x, y, z) = 0$$